1. Let $0 < \delta < \pi$, and define the 2π periodic function f by

$$f(x) = \begin{cases} 1, & \textit{if } |x| \le \delta \\ 0, & \textit{if } \delta < |x| \le |\pi| \end{cases}$$

   (a) Compute the Fourier coefficients of f.

   (b) Show that
$$\sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}.$$

   (c) Show that
$$\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2\delta} = \frac{\pi - \delta}{2}.$$

   (d) Show that
$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

Ans:

i $a_0 = \delta/\pi, a_n = 2\sin n\delta/n\pi, b_n = 0$.

ii Evaluate the Fourier series at $x = 0$.

iii Use Parseval's identity.

iv The idea of the solution is as follows: For a sufficiently large integer N, we can approximate the integral

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx \approx \int_0^{N} \left(\frac{\sin x}{x}\right)^2 dx.$$

And we can approximate the second integral using a Riemann sum. In fact, let m be a sufficiently large positive integer, we can partition the interval $[0, N]$ into Nm sub-intervals of equal length $\frac{1}{m}$, and the integral is then approximated by the corresponding (right) Riemann sum:

$$\begin{aligned} S_{N,m} &= \frac{1}{m} \sum_{n=1}^{mN} \left(\frac{\sin \frac{n}{m}}{\frac{n}{m}}\right)^2 \\ &= \sum_{n=1}^{mN} \frac{\sin^2 n\delta_m}{n^2\delta_m}, \end{aligned}$$

where $\delta_m = \frac{1}{m}$. Next, we denote S_m to be the sum:

$$S_m = \lim_{N\to\infty} S_{N,m} = \sum_{n=1}^{\infty} \frac{\sin^2 n\delta_m}{n^2\delta_m} = \frac{\pi - \delta_m}{2}.$$

We will will show that

$$\frac{\pi}{2} = \lim_{m\to\infty} S_m = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx,$$

the desired formula thus follows. In order to show the above equality, note that

$$\begin{aligned}&\left|S_m - \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx\right| \\ \le&|S_m - S_{N,m}| + \left|S_{N,m} - \int_0^N \left(\frac{\sin x}{x}\right)^2 dx\right| + \left|\int_N^\infty \left(\frac{\sin x}{x}\right)^2 dx\right|.\end{aligned}$$

Let $\epsilon > 0$, what we need to do is to choose m and N carefully so that each term of the right hand side of the above inequality is less than ϵ.
We begin with the last term. Note that

$$0 \le \int_N^\infty \left(\frac{\sin x}{x}\right)^2 dx \le \int_N^\infty \frac{1}{x^2} = \frac{1}{N^2},$$

which is $< \epsilon$ if

$$N > \epsilon^{-1/2}. \tag{1}$$

We next deal with the first term, if $m \ge 1$, and N' is a positive integer to be chosen, then

$$\begin{aligned}&|S_m - S_{N',m}| \\ \le& \sum_{n=N'm+1}^{\infty} \frac{\sin^2 n\delta_m}{n^2\delta_m} \\ =& \sum_{k=N'}^{\infty} \sum_{n=km+1}^{(k+1)m-1} \frac{\sin^2 n\delta_m}{n^2\delta_m} \\ \le& \sum_{k=N'}^{\infty} \sum_{n=km+1}^{(k+1)m-1} \frac{1}{(km)^2\delta_m} \\ =& \sum_{k=N'}^{\infty} \frac{1}{k^2}.\end{aligned}$$

Since $\sum_{k=1}^\infty \frac{1}{k^2} < \infty$, we can choose N' so that the last sum is $< \epsilon$, in other words, we can take

$$N \ge N' \tag{2}$$

It remains the middle term. Fixing an integer N satisfying the condition (1) and (2).

Since the function $\frac{\sin^2 x}{x^2}$ is continuous on $[0, N]$, it is integrable on $[0, N]$. Therefore, we can find a $\delta > 0$ such that for any partition P of $[0, N]$, with $||P|| < \delta$ and any Riemann sum S with respect to the partition P, we have

$$\left| S - \int_0^N \left(\frac{\sin x}{x} \right)^2 dx \right| < \epsilon.$$

Hence, if we choose $M = 1/\delta$, then

$$\left| S_{N,m} - \int_0^N \left(\frac{\sin x}{x} \right)^2 dx \right| < \epsilon$$

whenever $m \geq M$.
To conclude, we have shown that for each $\epsilon > 0$, there exists a positive number M such that

$$\left| S_m - \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx \right| < 3\epsilon$$

for all $m \geq M$. This finishes the analysis.